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On the Hausdorff Dimension of Fractal Attractors

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We consider such mappings $x_{n+1} = F(x_n)$ of an interval into itself for which the attractor is a Cantor set. For the same class of mappings for which the Feigenbaum scaling laws hold, we show that the Hausdorff dimension of the attractor is universally equal to $D = 0.538\ldots$.

KEY WORDS: One-dimensional mappings; Cantor sets, attractors, Hausdorff dimension.

1. INTRODUCTION

One of the basic properties of a strange attractor[1] is its Hausdorff dimension. For any bounded set of points in $\mathbb{R}^n$, it is defined[2] via the number N of small n-balls of size l needed to cover the set. If this number increases for $l \to 0$ like

$$N(l) \sim \text{const.}\, l^{-D}, \tag{1}$$

then D is called the Hausdorff dimension of the set. (More general cases are discussed in Ref. 2, but will not be needed here).

A strange attractor being topologically the product of a Cantor set with some $\mathbb{R}^p$, its Hausdorff dimension is in general noninteger. In Refs. 3 and 4, it has been estimated for several models existing in the literature.

In the present paper, we shall calculate the dimension of what can be considered the simplest strange attractor, namely the attractor in those nonlinear but smooth mappings

$$x_{n+1} = F(x_n) \tag{2}$$

for which it is a Cantor set. These mappings have remarkable universal scaling properties which were discussed extensively by Feigenbaum[5,6] (see also Refs. 7–12). These properties are very reminiscent of critical phenom-

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ena. Since fractal dimensions are closely related to anomalous dimensions[2,13], our findings add to this analogy. Consequently, we shall follow Collet and Tresser[10], and call such mappings "critical".

To be specific, consider a mapping of $[-b, b]$ into itself with

$$F(x) = b \cdot f(x), \qquad b > 0, \tag{3}$$

and f satisfying the conditions

— $f(0) = 1$

— $f(x)$ has a unique maximum at $x = 0$

— f is differentiable with $f' > 0$ for $x < 0$ and $f' < 0$ for $x > 0$.

— f'' exists around $x = 0$, and $f''(0) \neq 0$.

A typical example of such a function is shown in Fig. 1.

For small values of b, this mapping has one stable fixed point. When increasing b beyond some value b_1, this becomes unstable and a stable cycle of period 2 appears. This becomes again unstable at $b = b_2$, giving way to a period-4 attractor which for $b > b_3$ bifurcates into a period-8 attractor, etc. At the accumulation point b_{cr} of these bifurcation points b_i, the attractor becomes nonperiodic. For the logistic equation[2] $f(x) = 1 - 2x^2$, e.g., this happens for[5] $b_{\text{cr}} = 0.8370051\ldots$. At still larger values of b, one finds again periodic attractors (of period > 2) which bifurcate in a similar way. Again for $f(x) = 1 - 2x^2$, one finds, e.g., the accumulation point of the cycles with period 3×2^k at[5] $b_{\text{cr}} = 0.9433499\ldots$.

The interest in these "critical" mappings results from the universal properties found by Feigenbaum. He showed the following:

(i) The distance between successive bifurcation points scale like

$$b_{k+1} - b_k \underset{k\to\infty}{\sim} \delta^{-k} \tag{4}$$

with $\delta = 4.66920\ldots$ for all mappings satisfying the above conditions. These conditions are, however, much too stringent as this scaling law has been observed in many other systems as well.[8,9]

(ii) For $b = b_{\text{cr}}$ and for $x \approx 0$, the iterations

$$F^{(n)}(x) = \underbrace{F(F(\ldots F(x)\ldots))}_{n \text{ times}} \tag{5}$$

are self-similar for $n \to \infty$:

$$F^{(2n)}(x) \approx -(1/\alpha)F^{(n)}(\alpha x) \qquad (\text{for } x \approx 0, \quad n \to \infty) \tag{6}$$

with $\alpha = 2.50290\ldots$ being again a universal constant. The function

$$g(x) = \lim_{n\to\infty} (-\alpha)^n F^{(2^n)}(x/\alpha^n) \tag{7}$$

[2] The standard logistic equation is related to this by trivial manipulations.

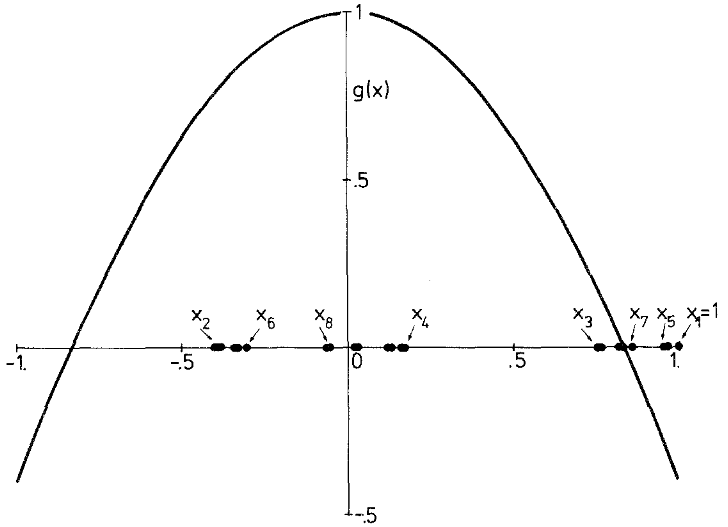

Fig. 1. The universal scaling curve $g(x)$ (calculated from the parametrization given in Ref. 6). The points on the x axis represent the attractor, with $x_{k+1} = g(x_k)$.

thus satisfies the exact scaling relation

$$g(g(x)) = -(1/\alpha)\, g(\alpha x) \tag{8}$$

Rescaling it according to

$$g(x) \to [1/g(0)]\, g(g(0)x) \tag{9}$$

we check easily that the rescaled function satisfies $g(0) = 1$ in addition to Eq. (8). It is indeed universal within the above class, and is plotted in Fig. 1.

In Section 2, we shall show that the known[6] properties of $g(x)$ allow a straightforward estimation of the Hausdorff dimension of the attractor of

$$x_{n+1} = g(x_n) \tag{10}$$

We shall find the exact bounds

$$0.53763 < D < 0.53854 \tag{11}$$

Since the attractors for different mappings are only locally universal, one cannot conclude immediately that D is universal.

In Section 3 we shall first present heuristic arguments that D is nevertheless universal. After that, we shall verify this for various critical mappings of the above class by brute force, by covering the attractor by small intervals, and fitting $N(l)$ by Eq. (1).

2. THE MAPPING DEFINED BY THE SCALING FUNCTION

Let us first construct the attractor of $x_{n+1} = g(x_n)$. Assume that

$$x_1 = g(0) = 1 \tag{12}$$

Then clearly $x_2 = -1/\alpha$, and the whole attractor lies in $[x_2, x_1]$ (see Fig. 1). Similarly, any point x_k is mapped by $g^{(k)}$ into

$$x_{2k} = -(1/\alpha)x_k, \qquad k = 1, 2, 3, \ldots . \tag{13}$$

Thus the set $\{x_k; k \text{ even}\}$ lies in $[x_2, x_4]$ and is exactly similar to the whole set $\{x_k\}$, but scaled down by a factor $-\alpha^{-1}$. The set $\{x_k; k \text{ odd}\}$ lies in the disjoint interval $[x_3, x_1]$. Since $g(x)$ is strictly monotonic and smooth on this interval, and since the set of even x_k is obtained from the set of odd x_k by applying g, the distribution of points x_k in $[x_3, x_1]$ is qualitatively similar to the pattern on $[x_2, x_4]$ (see Fig. 1), but slightly distorted.

The set $\{x_k\}$ separates thus into two subsets, one of which is exactly similar to it, and the other approximately. Repeating the same reasoning, we see that each subset is itself composed of two sub-subsets, each of which is (qualitatively) similar to the whole set and is composed of two sub-sub-subsets . . . etc.

This shows in particular that the points x_{2k+1} approach x_1 for $k \to \infty$, and thus x_1 belongs to the attractor. The whole attractor evidently consists of the set $\{x_k\}$.

Let us now cover the interval $[-1, 1]$ by small intervals of length l. Denoting by $N_{[2,4]}(l)$ and $N_{[3,1]}(l)$ the numbers of intervals needed to cover the points in $[x_2, x_4]$ and $[x_3, x_1]$, respectively, the above considerations show that

$$N_{[2,4]}(l) = N(\alpha l) \tag{14}$$

and

$$N_{[2,4]}(|g'(x_1)|l) < N_{[3,1]}(l) < N_{[2,4]}(|g'(x_3)|l) \tag{15}$$

In the last line we have used the fact that $g''(x) < 0$ in $[x_3, x_1]$. Assuming Eq. (1) to hold on each subinterval (with the same D!), we find

$$\frac{1}{\alpha^D} + \frac{1}{(\alpha|g'(x_1)|)^D} < 1 < \frac{1}{\alpha^D} + \frac{1}{(\alpha|g'(x_3)|)^D} \tag{16}$$

Calculating the x_k and $g'(x_k)$ from the parametrization given in Ref. 6, we obtain thus

$$0.5245 < D < 0.5544 \tag{17}$$

Successively more tight bounds are obtained by dividing the interval $[x_3, x_1]$ into 2^k subintervals, and using the approximate similarity of each of them to a subinterval of $[x_2, x_4]$. The next more stringent bound is, e.g., obtained by writing

$$N(l) = N_{[2,4]}(l) + N_{[3,7]}(l) + N_{[5,1]}(l) \tag{18}$$

with

$$N\left(\alpha^2 |g'(x_7)| l\right) < N_{[3,7]}(l) < N\left(\alpha^2 |g'(x_3)| l\right) \tag{19}$$

$$N_{[3,1]}\left(\alpha |g'(x_1)| l\right) < N_{[5,1]}(l) < N_{[3,1]}\left(\alpha |g'(x_5)| l\right) \tag{20}$$

and assuming again Eq. (1) for all subintervals.

The resulting bounds are $0.53313 < D < 0.54374$. Dividing $[x_3, x_1]$ into four pieces, we obtain $0.53646 < D < 0.53964$, while dividing it into eight subsets, we get finally Eq. (11).

3. UNIVERSALITY

As noted by Feigenbaum,[5,6] the attractor cannot be globally universal, although it is so locally around $x = 0$. Although the Hausdorff dimension is in some sense a global property, we propose that it is nevertheless universal.

Consider some small neighborhood I of $x = 0$, and an arbitrary point $x \notin I$ of the attractor. After some finite number k of mappings, x will be mapped onto a point in I. When this happens the first time, the function $F^{(k)}$ has a nonvanishing derivative at x. The piece of the attractor near x is thus obtained from the universal part around $x = 0$ by the monotonic inverse mapping $[F^{(k)}]^{-1}$, and is thus also approximately universal except for an x-dependent scale factor.

This suggests that the Hausdorff *measure* of the attractor [the constant in front of the exponential in Eq. (1)] is not universal, while the dimension is universal.

In order to check this, we studied numerically three different critical mappings which were also studied in Ref. 5: (1) the logistic equation $F(x) = b \cdot (1 - 2x^2)$ with $b = 0.8370051\ldots$ (limit point of period 2^k cycles), (2) the same equation with $b = 0.9433499\ldots$ (limit point of period $3 \cdot 2^k$ cycles), and (3) $F(x) = b \cdot x(1 - x^2)$ with $b = 2.302283\ldots$ (limit of period 2^k cycles). Notice that the last function is not of the general type demanded above, but can be brought to it by trivial manipulations.

Starting with x_1 equal to the maximum of $F(x)$, the attractor is in all three cases just the set of iterates x_k. [The proof goes as in the case where $F(x) = g(x)$.] We divided the interval $[-1, 1]$ into equal bins of length $l = 0.01 \times 2^{-n}$ and counted the number N of bins containing at least one x_k, for $1 \leqslant k \leqslant k_{\max}$. We did this for n up to 14. We found that N grew with $k_{\max}$ up to $k_{\max} \approx 65{,}000$ (for $n = 14$), but stayed constant beyond that (we checked this by iterating up to $k_{\max} = 10^5$). Plotting $\ln N$ against

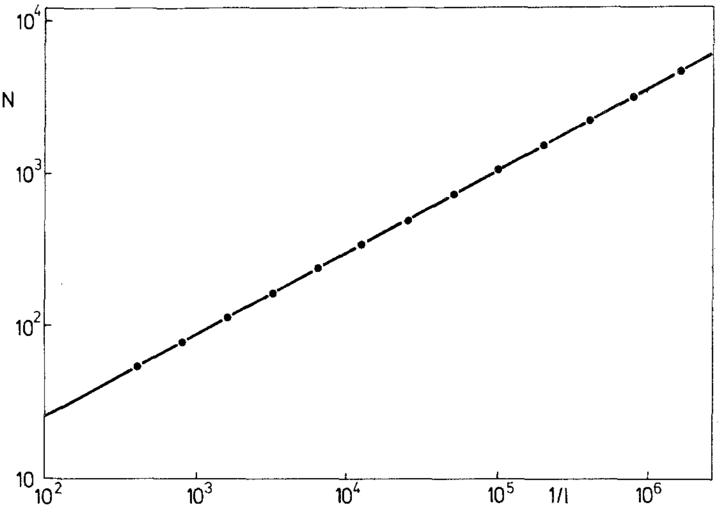

Fig. 2. The number $N(l)$ of bins of length l needed to cover the attractor of the logistic equation at the limit point of period 2^k cycles. By shifting the origin of the binning, $N(l)$ changes by amounts which are smaller than the sizes of the dots.

$\ln(1/l)$, we got perfectly straight lines, at least for $l \lesssim 0.0025$ (see Fig. 2). The slopes of these lines are just D. There are small unsystematic deviations from this straight line behavior which change randomly if we shift the origin of the binning by random numbers. These small deviations allow us to estimate "statistical" errors in a phenomenological way. Possible systematic errors are not taken into account in this way. They could arise if our bins were not yet small enough, and they are not easy to estimate reliably (as in all numerical simulations of critical phenomena!). But they seem to be negligible if we use for a fit only binnings with $l \leqslant 3 \times 10^{-4}$, as indicated by the linearity of the plots down to much larger l's.

We found in this way

$$D = \begin{cases} 0.5381 \pm 0.0006 & \text{for case (1)} \\ 0.5388 \pm 0.002 & \text{for case (2)} \\ 0.5388 \pm 0.002 & \text{for case (3)} \end{cases} \tag{21}$$

in agreement with the universality hypothesis. As anticipated, the Hausdorff *measures* were observed to be nonuniversal.

4. COMMENTS

Let us add some remarks about the estimates of D presented in Refs. 3 and 4. There it was shown that D is related to the Lyapunov characteristic exponents, for a large number of strange attractors. In our cases, there is but one characteristic exponent which is easily seen to vanish for critical mappings. Applying naïvely the formula given by Mori,[3] we would predict absurdly $D = 1$.

One reason for this failure might be the observation of Yorke (cited in Ref. 4) that this connection between D and characteristic exponents need not to hold for x-dependent Jacobians $\|\partial F_i(x)/\partial x_k\|$.

Anyhow, characteristic exponents are not useful to describe the asymptotic behavior of $d|F^{(n)}|/dx$ for critical mappings, being too crude a measure for the sensitivity to initial conditions. For critical mappings, $d|F^{(n)}|/dx$ behaves extremely nonuniformly both with respect to n and to x. In a suitably averaged sense, however, one finds it to increase like some universal power of n, as the analogy with critical phenomena would have suggested. Details will be given elsewhere.[14]

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